Rates of Change and Differentiation

Before You Watch

Before watching this video, make sure you've seen **Introduction to Calculus** first, as this video follows on from the introduction.

This topic also discusses some of the concepts that were introduced in **Linear Equations**, so consider watching that one as well, then come back.

The Video Content

In the Introduction to Calculus, we considered a car moving, and we saw that its speed at an instant is given by:

dx / dt

where:

dx means a tiny change in position

and:

dt is a tiny change in time.

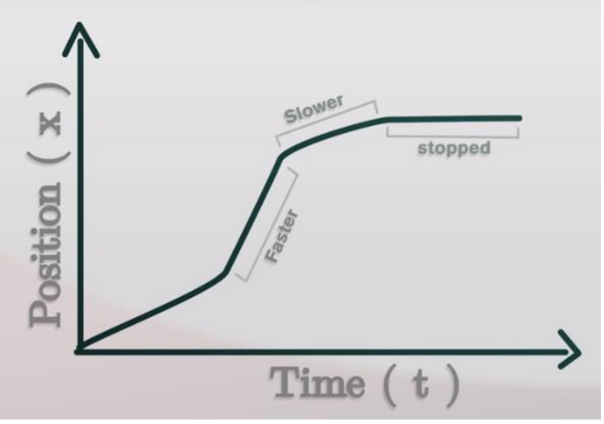
So, speed tells us how two things change together. It tells us how position changes as time changes.

This is what we call a *rate of change*.

Rates of change are all around us. Any time we compare two values and see how they change together, we are considering a rate of change. How much fuel does my car consume to drive 100 km? That's measuring a change in fuel against a change in position. How long until my beer is cold? That's measuring a change in temperature against a change in time. You get the idea.

Going back to the car problem, let’s look at a graph of it.

The car’s *position* is up the vertical axis, and the *time* is along the horizontal axis.



As the car moves across the graph, time goes by, and the position increases:

- if the car goes faster, the position changes more as time passes, so the line is steeper
- if the car slows down, the line is less steep
- if the car stops, then position doesn’t change but time does, so the line is horizontal.

As you can see, the rate of change of the car’s position versus time – also known as the car's speed – is related to the gradient.

Remember that *gradient* is *rise* over *run*?

In this example, the rise over any bit of the line is a change in position, and the run is a change in time.

So rise over run is:

change in position / change in time.

But we already know that change in position is dx and change in time is dt.

Therefore gradient is the rate of change – they are the same thing!
In this example, the rate of change is speed.

Did you know?

In this example the y axis is position, and the x axis is time, so the gradient is:

change in position / change in time = speed

Another way of thinking about this is to consider some example units of measurement. In this instance, position can be measured in metres, and time measured in seconds, so a suitable unit for the gradient is metres per second.

Or, position could be measured in kilometres, and time in hours, so the gradient would be measured in kilometres per hour.

At university you will come across many graphs and be asked to interpret the meaning of the gradient. The method, however, remains the same:

gradient = change in y axis / change in x axis

The units for this gradient can always be thought of as:

[units of y axis] per [units of x axis]

Let's say a car is travelling at 60 km/hr. The position of the car is given by:

$$x = 60t$$

where t is the time in hours.

At one hour it's 60 km away, at two hours it's 120 km away and so on.

The speed is 60, so:

$$dx \,/\, dt = 60$$

The process of going from an equation for position, x, to an equation for speed, dx / dt, is called *differentiation*.

Therefore, to find the rate of change, which is the gradient, we need to differentiate.

It is important to remember that:

- *differentiation* is finding the *rate of change* between two values
- the *rate of change* is represented graphically by the *gradient*.

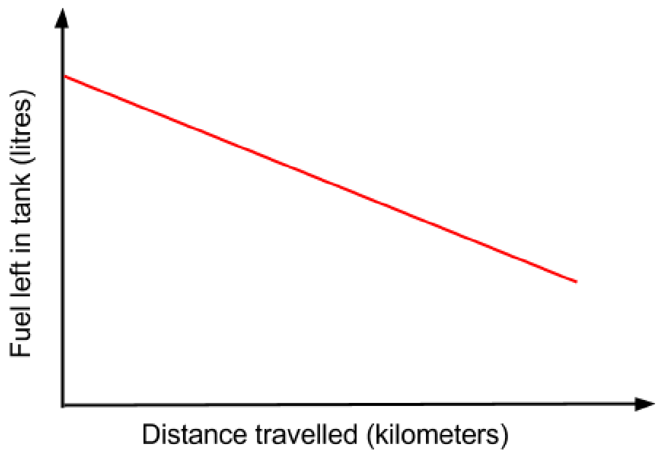
In this example, it might not seem necessary to differentiate. After all, we know the speed is 60 km/hr. But what happens if the car is accelerating?

In this situation, the speed, the gradient, is constantly changing. Let’s think about calculating the gradient. If we pick any two points in time, we can find the average speed over this time, but that won’t be the correct speed at either time.

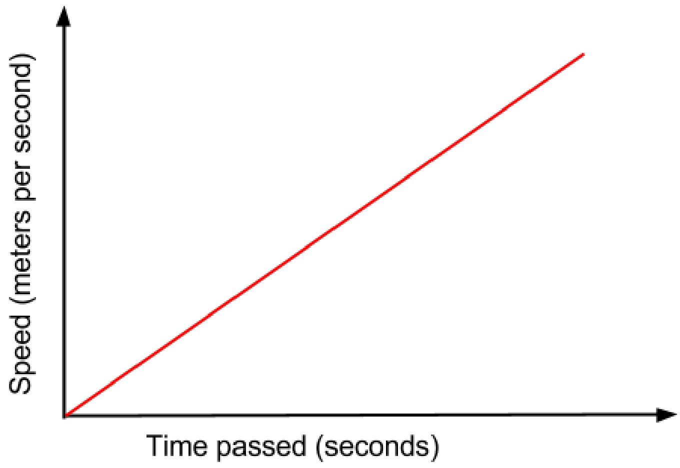
In this example, the formula of rise over run will fail. Therefore, to find the gradient at a point, we need to find the gradient of the tangent at that point. Mathematically, this is what differentiation is. We will look at this situation in the next topic.

Some Practice Questions

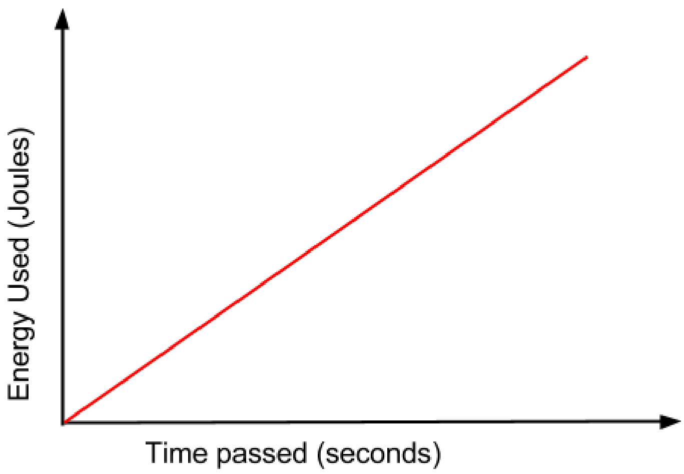
1. Read the graph below, and determine the correct *units* (not the value, the units) for the *slope (*or rate of change) of the graph.



2. Read the graph below, and determine the correct *units* (not the value, the units) for the *slope (*or rate of change) of the graph.



3. Read the graph below, and determine the correct *units* (not the value, the units) for the *slope (*or rate of change) of the graph.



Answers

1. Litres per kilometre (L/km). This rate of change is often used to measure the fuel economy of cars. Usually the distance is measured in hundreds of kilometres, so the unit would be litres per hundred kilometres.

2. Metres per second per second (m/s/s) *or* metres per second squared (m/s^2 *or* ms^{-2}). This is the standard scientific unit for measuring acceleration. When you think about it, we are measuring how speed changes over time, and that's acceleration.

3. Joules per second (J/s). This is also known as Watts, and is the standard unit of power, commonly used to measure, for example, the power of your car, heater or microwave. Power is therefore the rate of change between energy and time; it measures how much energy is used over time. Another way of putting it is that power measures how much time it takes to use energy.

Now What?

By now you will be familiar the basics of calculus, the meaning of rates of change, and why we are interested in rates of change. You should also understand the concept of differentiation, which is the mathematical process of going from one formula that relates two variables (such as position and time) to another formula that gives the rate of change between those two variables (such as the rate of change between position and time, also known as speed).

The next step is to learn how to differentiate. In the next topic, **Differentiation of Polynomials**, you'll be shown how to differentiate the group of equations known as polynomials.

But When Am I Going To Use This?

Calculus is the mathematical study of how things change relative to one another. For instance, velocity (or speed) is a change of position over a change in time, and acceleration is a change in velocity over a change in time – so any motion is studied using calculus. Other examples include the flow of water through pipes over time, or changing commodity prices against demand. Because change is everywhere, the potential applications for calculus are endless, particularly in engineering and science. Calculus is necessary knowledge for any degree related to engineering or science.

Other Links

Maths is Fun has a great page that takes you through a simple problem which highlights the need for calculus to discuss changes happening around us. It then continues to explore the main two areas of calculus, differentiation and integration, and provides regular questions to test your understanding.

- https://www.mathsisfun.com/calculus/introduction.html

IntMath gives a bit of historical perspective to explain the sometimes confusing notation that is used in calculus, discussing how it is the mixed product of two mathematicians working independently. It also provides some excellent examples of applications of calculus that are in common use today, as well as helpful applets to understand both differential and integral calculus.

- http://www.intmath.com/calculus/calculus-intro.php

The **Khan Academy** has a comprehensive set of video tutorials covering a wide range of mathematical topics, as well as questions to test your knowledge. This content explains the historical development of calculus, and is also an excellent introduction to differential calculus and the concepts it is based around. From here you can further investigate differential calculus.

- https://www.khanacademy.org/math/differential-calculus/taking-derivatives/intro_differential_calc/v/newton-leibniz-and-usain-bolt

Patrick JMT (Just Maths Tutorials) has an extensive set of video tutorials covering a large range of mathematical concepts. This video introduces and explains the concept of a limit to help develop your understanding of this idea.

- https://patrickjmt.com/what-is-a-limit-basic-idea-of-limits/